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Aspects of monopole operators in $\mathcal{N}=6$ Chern-Simons theory

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ABSTRACT: We study local operators of $U(N) \times U(N)$ $\mathcal{N}=6$ Chern-Simons-matter theory including a class of magnetic monopole operators. To take into account the interaction of monopoles and basic fields for large Chern-Simons level k , we consider the appropriate perturbation theory in $\frac{1}{k}$ which reliably describes small excitations around protected chiral operators. We also compute the superconformal index with some simple monopole operators and show that it agrees with the recent result obtained from localization. For this agreement, it is crucial that excitations of gauge fields and some matter scalars mix, which is described classically by odd dimensional self-duality like equations.

KEYWORDS: AdS-CFT Correspondence, Chern-Simons Theories, M-Theory

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1 Introduction

Recent studies of AdS₄/CFT₃ states that a class of Chern-Simons-matter theories provide holographic descriptions of M-theory. See, among others, [1–4]. An important ingredient for understanding M-theory in this setting is the magnetic monopole operator [5], creating the gauge theory duals of Kaluza-Klein modes along the ‘eleventh direction’ beyond type IIA backgrounds. For recent studies of monopole operators, see [6–13]. The $\mathcal{N}=6$ Chern-Simons-matter theory [4] is an ideal model to study this subject.

Generically, operators (or states via radial quantization) involving monopole operators can be understood only after taking large interactions into account even if the coupling constant $\frac{1}{k}$ is small, where k is the Chern-Simons level. A simple argument goes as follows. For simplicity, consider a subsector of $U(N) \times U(N)$ $\mathcal{N}=6$ theory consisting of the gauge fields A_μ, \tilde{A}_μ and complex scalars ϕ_I . The argument generalizes to other theories. The action takes the form

$$\int \text{tr} \left[\frac{k}{4\pi} \left(AdA - \frac{2i}{3} A^3 \right) - \frac{k}{4\pi} \left(\tilde{A}d\tilde{A} - \frac{2i}{3} \tilde{A}^3 \right) - D_\mu \phi_I D^\mu \bar{\phi}^I - \frac{1}{k^2} V(\phi) \right], \quad (1.1)$$

where $V(\phi)$ is a potential which is of sixth order in ϕ_I . The equations of motion for A_μ, \tilde{A}_μ are

$$\frac{k}{2\pi} \star F_\mu = i \left(D_\mu \bar{\phi}^I \phi_I - \bar{\phi}^I D_\mu \phi_I \right), \quad \frac{k}{2\pi} \star \tilde{F}_\mu = i \left(\phi_I D_\mu \bar{\phi}^I - D_\mu \phi_I \bar{\phi}^I \right). \quad (1.2)$$

With magnetic monopoles, the left hand sides on a spatial 2-sphere integrate to $\mathcal{O}(k)$ numbers due to the flux quantization, which requires the scalars to be $\mathcal{O}(k^{\frac{1}{2}})$. With scalars at this order, one generically finds that both the kinetic term and the potential $\frac{1}{k^2} V(\phi)$ contribute to the scalar equation of motion in same orders, implying that conventional perturbative approach cannot be valid.¹

¹An important exception is the case in which $V(\phi)$ vanishes, which corresponds to chiral operators. We shall consider these exceptional cases as well in this paper.

In this paper, we investigate the correct perturbation theory at large k appropriate for studying the spectrum of these operators. We start by studying the classical field theory. We first find the lowest energy configurations solving all equations of motions in the interacting theory including (1.2), for a class of magnetic fluxes turned on. See section 2 for the details on the monopole operators we consider. By studying the fluctuation of fields in this exact background, we find the desired perturbation theory in $\frac{1}{k}$. In this paper, we mainly study the ‘free theory’ limit ignoring subleading interactions, leaving more elaborate study as a future work. Quantizing the modes in the free theory, one can immediately calculate the partition function for these operators.

The spectrum that we obtain in this free theory is subject to change as one considers interactions suppressed by $\frac{1}{k}$. However, spectrum of certain states preserving supersymmetry can be stable against interaction at least in the ’t Hooft limit, which takes $N, k \rightarrow \infty$ with $\lambda = \frac{N}{k}$ finite [14]. Such states are counted by the superconformal index [15]. Recently the superconformal index for the $\mathcal{N}=6$ Chern-Simons-matter theory has been computed and studied [9] in the sectors containing monopole operators, generalizing the earlier work [14]. In [9], the index is computed by applying localization technique to the path integral for the index. This calculation involves deforming the theory in a way that the index is not changed. Although this is a standard method, it should be illuminating if one can understand the same quantity directly from the $\mathcal{N}=6$ Chern-Simons-matter theory itself, at least for large k . We use our free theory to reproduce this index for a class of monopoles.

Monopole operators studied in [9] can be classified into two classes, according to their representations of $U(N) \times U(N)$ gauge group. Firstly, a monopole operator can be in a conjugate representation of the two $U(N)$ gauge groups. Such monopole operators can combine with basic field operators to form gauge invariant chiral operators, in the sense that their dimensions are given by the R-charge. These chiral operators are relatively well understood [6, 7, 10, 11]. It is also in this sector that we can find exact classical solutions with lowest energy. Secondly, it was shown in [9] that monopole operators in non-conjugate representations of two $U(N)$ *should* exist. This claim was solidly supported by a very detailed matching between the large N indices from gauge theory and supergravity. In this case, the lowest energy states come with nonzero spatial angular momenta, which is harder to study than the chiral operators in the first class. We do not know yet how to analyze this sector using the approach of this paper, and leave this problem as a future work.

The rest of this paper is organized as follows. In section 2 we provide the classical analysis of the field theory on $S^2 \times \mathbb{R}$, in the presence of magnetic fluxes. By studying all small fluctuations of charged modes and quantizing them, we calculate the superconformal index, which agrees with the result of [9]. We also briefly discuss open problems with monopole operators which are not considered in detail in this paper. In section 3 we conclude with a few comments.

Note. While we were finalizing this draft, [16] appeared with some overlap with our work.

2 Spectrum with monopole operators

The $\mathcal{N} = 6$ Chern-Simons-matter theory is often conveniently described by decomposing the fields in $\mathcal{N} = 2$ supermultiplets. In particular, this is useful for us since we shall consider the superconformal index of this theory in section 3, which uses $\mathcal{N} = 2$ supersymmetry only. In the notation of [9] (which closely follows [17]), the matter fields decompose to four chiral multiplets $(A_a, \psi_{a\alpha})$, $(B_{\dot{a}}, \chi_{\dot{a}\alpha})$ (with $a, \dot{a} = 1, 2$) in bifundamental and anti-bifundamental representations of $U(N) \times U(N)$, respectively. Table 1 summarizes the global charges of fields. $h_{1,2,3}$ are three Cartans of $SO(6)$ R-symmetry, and h_4 is the ‘baryon-like’ $U(1)_b$ charge. The action on \mathbb{R}^{2+1} is given as follows:

$$\mathcal{L} = \mathcal{L}_{\text{CS}} + \mathcal{L}_m, \quad (2.1)$$

where the Chern-Simons term is given by

$$\mathcal{L}_{\text{CS}} = \frac{k}{4\pi} \text{tr} \left(A \wedge dA - \frac{2i}{3} A^3 + i\bar{\lambda}\lambda - 2D\sigma \right) - \frac{k}{4\pi} \text{tr} \left(\tilde{A} \wedge d\tilde{A} - \frac{2i}{3} \tilde{A}^3 + i\tilde{\lambda}\tilde{\lambda} - 2\tilde{D}\tilde{\sigma} \right), \quad (2.2)$$

and (overbar for scalars denotes Hermitian conjugation)

$$\begin{aligned} \mathcal{L}_m = \text{tr} \left[& -D_\mu \bar{A}^a D^\mu A_a - D_\mu \bar{B}^{\dot{a}} D_\mu B_{\dot{a}} - i\bar{\psi}^a \gamma^\mu D_\mu \psi_a - i\bar{\chi}^{\dot{a}} \gamma^\mu D_\mu \chi_{\dot{a}} \right. \\ & - (\sigma A_a - A_a \tilde{\sigma}) (\bar{A}^a \sigma - \tilde{\sigma} \bar{A}^a) - (\tilde{\sigma} B_{\dot{a}} - B_{\dot{a}} \sigma) (\bar{B}^{\dot{a}} \tilde{\sigma} - \sigma \bar{B}^{\dot{a}}) \\ & + \bar{A}^a D A_a - A_a \tilde{D} \bar{A}^a - B_{\dot{a}} D \bar{B}^{\dot{a}} + \bar{B}^{\dot{a}} \tilde{D} B_{\dot{a}} \\ & - i\bar{\psi}^a \sigma \psi_a + i\psi_a \tilde{\sigma} \bar{\psi}^a + i\bar{A}^a \lambda \psi_a + i\bar{\psi}^a \tilde{\lambda} A_a - i\psi_a \tilde{\lambda} \bar{A}^a - iA_a \tilde{\lambda} \bar{\psi}^a \\ & \left. + i\chi_{\dot{a}} \sigma \bar{\chi}^{\dot{a}} - i\bar{\chi}^{\dot{a}} \tilde{\sigma} \chi_{\dot{a}} - i\chi_{\dot{a}} \lambda \bar{B}^{\dot{a}} - iB_{\dot{a}} \tilde{\lambda} \bar{\chi}^{\dot{a}} + i\bar{B}^{\dot{a}} \tilde{\lambda} \chi_{\dot{a}} + i\bar{\chi}^{\dot{a}} \tilde{\lambda} B_{\dot{a}} \right] + \mathcal{L}_{\text{sup}}. \quad (2.3) \end{aligned}$$

\mathcal{L}_{sup} contains scalar potential and Yukawa interaction obtained from a superpotential

$$W = -\frac{2\pi}{k} \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \text{tr}(A_a B_{\dot{a}} A_b B_{\dot{b}}). \quad (2.4)$$

σ, λ, D and $\tilde{\sigma}, \tilde{\lambda}, \tilde{D}$ are auxiliary. The Lagrange multipliers D, \tilde{D} impose

$$\sigma = \frac{2\pi}{k} (A_a \bar{A}^a - \bar{B}^{\dot{a}} B_{\dot{a}}), \quad \tilde{\sigma} = \frac{2\pi}{k} (\bar{A}^a A_a - B_{\dot{a}} \bar{B}^{\dot{a}}). \quad (2.5)$$

$\lambda_\alpha, \tilde{\lambda}_\alpha$ are also given in terms of the matter fields as

$$\lambda = \frac{4\pi}{k} (\bar{\chi}^{\dot{a}} B_{\dot{a}} - A_a \bar{\psi}^a), \quad \tilde{\lambda} = \frac{4\pi}{k} (B_{\dot{a}} \bar{\chi}^{\dot{a}} - \bar{\psi}^a A_a). \quad (2.6)$$

This theory can be put on $S^2 \times \mathbb{R}$ via radial quantization. The basic procedure is to first consider the above theory on Euclidean \mathbb{R}^3 and regard the radius r in spherical coordinate system as the Euclidean time τ via $r = R e^\tau$, where R is an arbitrary dimensionful parameter. The metric on \mathbb{R}^3 is

$$ds_{\mathbb{R}^3}^2 = \left(\frac{r}{R} \right)^2 (R^2 d\tau^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2))^2 = \left(\frac{r}{R} \right)^2 ds_{\mathbb{R} \times S^2}^2. \quad (2.7)$$

fields	h_1	h_2	h_3	j_3	ϵ	h_4
(A_1, A_2)	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2})$	0	$\frac{1}{2}$	$\frac{1}{2}$
(B_1, B_2)	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2})$	0	$\frac{1}{2}$	$-\frac{1}{2}$
$(\psi_{1\pm}, \psi_{2\pm})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$\pm\frac{1}{2}$	1	$\frac{1}{2}$
$(\chi_{1\pm}, \chi_{2\pm})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$\pm\frac{1}{2}$	1	$-\frac{1}{2}$
A_μ, \tilde{A}_μ	0	0	0	$(1, 0, -1)$	1	0

Table 1. Charges of fields.

The metric $ds_{\mathbb{R} \times S^2}^2$ refers to a round 2-sphere with radius R . Due to the dilatation invariance of the theory, the prefactor $(\frac{r}{R})^2$ appearing in the action is canceled and one can rewrite the above action as a Euclidean theory defined on $\mathbb{R} \times S^2$. See, say, [9] for the details. One of the important changes is that scalars acquire the conformal mass $m^2 = \frac{1}{4R^2}$. Without losing generality, we set the radius of S^2 to $R = 1$ in the rest of this paper except for a few occasions which might cause confusions. (In all cases, factors of R can be restored by dimensional analysis.) Our final step is to go back to the Minkowskian theory on $\mathbb{R} \times S^2$ by a Wick rotation $t = -i\tau$. In section 2.1, we will sometimes use the Euclidean-like variables τ or r to take advantage of the Cartesian analysis on \mathbb{R}^3 in the Minkowskian theory. Still, one should bear in mind that t is real throughout this paper, or $r = Re^{it}$ is a phase up to a factor R which we set to 1. Same idea (i.e. $r \rightarrow r^{-1}$ under complex conjugation) appeared in [18].

In the radially quantized theory, one can consider configurations in which nonzero magnetic flux on spatial S^2 is turned on. From the representations of matter fields under $U(N) \times U(N)$, one finds that $\text{tr}F = \text{tr}\tilde{F}$ should be satisfied. The Kaluza-Klein momentum in the dual M-theory along the fiber circle of S^7/\mathbb{Z}_k is given by

$$P = \frac{k}{4\pi} \int_{S^2} \text{tr}F = \frac{k}{4\pi} \int_{S^2} \text{tr}\tilde{F} \tag{2.8}$$

in the gauge theory [4]. This, via Gauss' law constraint, turns out to be proportional to h_4 in table 1. Monopole operators create these fluxes on S^2 . In particular, we can embed $U(1)^N \times U(1)^N$ Dirac monopoles to $U(N) \times U(N)$ such that

$$\frac{1}{4\pi} \int_{S^2} F = \frac{1}{2} \text{diag}(n_1, n_2, \dots, n_N), \quad \frac{1}{4\pi} \int_{S^2} \tilde{F} = \frac{1}{2} \text{diag}(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_N). \tag{2.9}$$

Here $H \equiv (n_1, n_2, \dots, n_N)$ and $\tilde{H} \equiv (\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_N)$ are non-increasing integers.

2.1 Classical solutions

In this paper, we mainly consider monopole operators with $n_1 = \tilde{n}_1 = n > 0$ and other $n_i, \tilde{n}_i = 0$. Sectors with negative n can be obtained by parity. In subsection 2.3, we discuss the technical complications that we encounter for other cases, including the cases with $H \neq \tilde{H}$.

The matter fields carry positive scale dimensions and nonzero $U(1)_b$ charge h_4 , where the latter charge has to be balanced with the total magnetic flux through the Gauss' law.

The total $U(1)_b$ charge is given by the magnetic flux as

$$\frac{k}{2} \sum_{i=1}^N n_i = \frac{k}{2} \sum_{i=1}^N \tilde{n}_i = \frac{kn}{2} > 0. \quad (2.10)$$

Our strategy is to first obtain classical solutions which would account for the states with lowest energy (after quantization) for a given positive $U(1)_b$ charge given above, and then study the small fluctuations with higher energy. We will see that the latter modes can be treated perturbatively in $\frac{1}{k}$. From table 1, the lowest energy states with positive $U(1)_b$ charge are given by creating states only with operators $\bar{B}^{\dot{1}}, \bar{B}^{\dot{2}}$, or only with A_1, A_2 , in their s-waves. The two cases can be analyzed in a completely same way. They are annihilated by different combinations of supercharges. Here we consider the first sector only.

Turning on the gauge fields and $B_{\dot{a}}$, the equations of motion for A_μ, \tilde{A}_μ are given by

$$\frac{k}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = -i\sqrt{-g} (\bar{B}^{\dot{a}} D^\mu B_{\dot{a}} - D^\mu \bar{B}^{\dot{a}} B_{\dot{a}}) \quad (2.11)$$

where $\epsilon^{t\theta\phi} = 1$. Note that this equation takes the same form as that for the theory on \mathbb{R}^{2+1} , apart from the different metric, since the conformal mass term on $\mathbb{R} \times S^2$ does not contribute. The equation of motion for $B_{\dot{a}}$ on $S^2 \times \mathbb{R}$ is given by

$$\left(D^\mu D_\mu - \frac{1}{4} \right) B_a = \frac{4\pi^2}{k^2} \left[3(B_b \bar{B}^b)^2 B_a + 3B_a (\bar{B}^b B_b)^2 - 2B_b \bar{B}^c B_c \bar{B}^b B_a - 2B_a \bar{B}^b B_c \bar{B}^c B_b - 2B_b \bar{B}^b B_a \bar{B}^c B_c \right]. \quad (2.12)$$

Our solution has nonzero uniform magnetic fields on S^2 in the first $U(1)$ among $U(1)^N$ in each $U(N)$, and s-wave of $B_{\dot{a}}$ is nonzero in the 11 component, where the first (second) 1 denotes the first component in the anti bi-fundamental (bi-fundamental) of first (second) $U(N)$ group. The Gauss' law demands

$$B_{\dot{a}} = b_{\dot{a}} e^{-it/2} \quad (2.13)$$

where the complex constants $b_{\dot{a}}$ satisfy

$$|b_1|^2 + |b_2|^2 = \frac{kn}{4\pi}. \quad (2.14)$$

We also set $A_t = 0$. One finds that this solution also satisfies the scalar equation of motion (2.12) with the right hand side from the potential vanishing. The positive frequency $\omega = \frac{1}{2}$ of our solution implies that \bar{b}^a modes are to be regarded as creation operators after quantization. These states belong to protected short multiplets, which is easy to see as they are obtained by $SU(2)$ actions on states of the form $(b_1^\dagger)^{kn} |0\rangle$ [19].

We now investigate fluctuations of all fields around the above background in the leading order in $\frac{1}{k}$. The modes run over $A_a, B_{\dot{a}}, \psi_a, \chi_{\dot{a}}$ in the matter fields as well as the vector fields A_μ, \tilde{A}_μ which turn out to couple to some of the matter fields.

We start by considering bosonic fluctuations. We first consider the fluctuations δA_a in bifundamental of $U(N) \times U(N)$, which can be considered separately since they do not mix in

the leading order with other fields in the background with nonzero B_a . The mode $(\delta A_a)_{11}$ or $(\delta A_a)_{ij}$ do not couple to magnetic field or the background scalar B_a in the leading quadratic order, where $i, j = 2, 3, \dots, N$ are fundamental/anti-fundamental indices for $U(N-1) \times U(N-1)$. These are simply expanded with the spherical harmonics, which is the same as the weakly coupled theory without monopoles.

The modes $(\delta A_a)_{1i}$ or $(\delta A_a)_{i1}$, which couple to $\pm n$ units of magnetic charges, are expanded with monopole spherical harmonics. Monopole spherical harmonics are labeled by the total angular momentum $j = \frac{|n|}{2}, \frac{|n|}{2} + 1, \frac{|n|}{2} + 2, \dots$ and the Cartan $j_3 = m$. From the kinetic and the conformal mass terms, one obtains

$$\frac{d(\delta A_a)}{dt} \frac{d(\delta \bar{A}^a)}{dt} - \left[\left(j + \frac{1}{2} \right)^2 - \frac{n^2}{4} \right] \delta A_a \delta \bar{A}^a, \quad (2.15)$$

where δA_a denotes either $1i$ or $i1$ component. The leading contribution of this fluctuation is also present in the potential. In the potential, it may appear either by directly fluctuating A_a, \bar{A}^a which are explicit in (2.3), or via fluctuations of the composite fields $\sigma, \tilde{\sigma}$. The latter possibility yields no leading terms, quadratic in δA_a : $\delta\sigma, \delta\tilde{\sigma}$ coupling to A_a 's obviously starts from sextic fluctuations with coefficients $\frac{1}{k^2}$ while those coupling to $B_{\tilde{a}}$ always comes with a factor of the background fields $(\tilde{\sigma} B_{\tilde{a}} - B_{\tilde{a}} \sigma)$ or its conjugate which is zero since $\sigma = \tilde{\sigma}$ commute with background B_a . Thus we only study the direct fluctuations. One first finds a factor

$$- \frac{n^2}{4} \delta A_a \delta \bar{A}^a \quad (2.16)$$

from the term coupling to $\sigma, \tilde{\sigma}$. One should also consider the potential coming from the superpotential: the term relevant for the fluctuations δA_a is

$$- \frac{4\pi^2}{k^2} \text{tr} \left[\left(B_1 A_a B_2 - B_2 A_a B_1 \right) \left(\bar{B}^2 \bar{A}^a \bar{B}^1 - \bar{B}^1 \bar{A}^a \bar{B}^2 \right) \right]. \quad (2.17)$$

The contribution from this term is zero. This is easy to see, since by an $SU(2)$ internal rotation $b_{\tilde{a}}$ can be rotated to satisfy either $b_1 = 0$ or $b_2 = 0$. Since the superpotential is $SU(2)$ invariant and acquires nonzero contribution only when the two scalars are both nonzero, the vanishing of the fluctuation in this case is obvious.² Combining all, one finds that the last term in (2.15) and (2.16) cancel that the frequencies of these modes are given by

$$\omega^2 = \left(j + \frac{1}{2} \right)^2. \quad (2.18)$$

In particular, the spectrum $\omega = \pm \left(j + \frac{1}{2} \right)$ is crucial since some highest weight states, satisfying $j = j_3$, should saturate the BPS energy bound

$$\epsilon \geq R + j_3 = j_3 + \frac{1}{2} \quad (2.19)$$

where R is the R-charge of $\mathcal{N} = 2$ supersymmetry. This was also found in [9].

²In subsection 2.3, we will find a subtle contribution from the superpotential for backgrounds with more general monopoles.

We then turn to the fluctuations of the scalars δB_a in $1i$ and $i1$ component, where again the first and second indices are for the first and second of $U(N) \times U(N)$. (Again, ‘diagonal’ modes with 11 and ij are trivially expanded with spherical harmonics.) It is convenient to decompose

$$\delta B_a = z_a \delta\phi + \epsilon_{ab} \bar{z}^b \delta\varphi, \quad (2.20)$$

where we define $\epsilon_{12} = -\epsilon_{21} = 1$ and $b_a \equiv \sqrt{\frac{kn}{4\pi}} z_a$, satisfying $|z_1|^2 + |z_2|^2 = 1$. From its $SU(2)$ index structure, the mode $\delta\varphi$ does not directly couple to the background scalar b_a in the leading (quadratic) order and only couples to it via nonzero $\sigma, \tilde{\sigma}$. This is the same as the fluctuations δA_a above, leading to the same result (2.18).

Finally in the bosonic sector, we consider off-diagonal fluctuation $\delta\phi$. We denote by $\delta\phi^\pm$ the $i1$ and $1i$ components of the first and second gauge group, respectively. $\delta\phi^\pm$ couples to $\pm n$ units of fluxes. We also denote by $\phi_0 = \sqrt{\frac{kn}{4\pi}} e^{-it/2}$ the background field. By taking a glance at the off-diagonal components of (2.11) and (2.12), it turns out that one has to expand the off-diagonal A_μ, \tilde{A}_μ together with $\delta\phi$. Let us denote by δA_μ and $\delta \tilde{A}_\mu$ the $1i$ component in the adjoint of the first/second gauge group, respectively. From Gauss law one obtains

$$\begin{aligned} \star D\delta A &= \frac{2\pi}{k} \left[|\phi_0|^2 \delta A - i (\bar{\phi}_0 D\delta\phi^+ - d\bar{\phi}_0 \delta\phi^+) \right], \\ \star D\delta \tilde{A} &= \frac{2\pi}{k} \left[-|\phi_0|^2 \delta \tilde{A} - i (d\phi_0 \delta\bar{\phi}^- - \phi_0 D\delta\bar{\phi}^-) \right]. \end{aligned} \quad (2.21)$$

Inserting the value of $|\phi_0|^2$, we rewrite it as

$$\begin{aligned} \left(\star D - \frac{n}{2} \right) \delta A &= -\frac{2\pi i}{k} (\bar{\phi}_0 D\delta\phi^+ - d\bar{\phi}_0 \delta\phi^+), \\ \left(\star D + \frac{n}{2} \right) \delta \tilde{A} &= -\frac{2\pi i}{k} (d\phi_0 \delta\bar{\phi}^- - \phi_0 D\delta\bar{\phi}^-). \end{aligned} \quad (2.22)$$

$\delta A, \delta \tilde{A}$ are taken to be of same order as $\frac{1}{\sqrt{k}} \delta\phi^\pm$. D acts on fluctuations according to their charges: $D = d - iA^{(0)}$ on $\delta A, \delta \tilde{A}, \delta\phi^+$ and $D = d + iA^{(0)}$ on $\delta\phi^-$, where $A^{(0)}$ provides uniform $U(1)$ magnetic field with n units of flux. Expanding the scalar equation of motion, one finds

$$\begin{aligned} \left(D^\mu D_\mu - \frac{1}{4} \right) \delta\phi^+ + i\phi_0 (D^\mu \delta A_\mu) + 2i\partial_\mu \phi_0 \delta A^\mu &= 0 \\ \left(D^\mu D_\mu - \frac{1}{4} \right) \delta\bar{\phi}^- + i\bar{\phi}_0 (D^\mu \delta \tilde{A}_\mu) + 2i\partial_\mu \bar{\phi}_0 \delta \tilde{A}^\mu &= 0. \end{aligned} \quad (2.23)$$

Note that the potential does not contribute since it vanishes for a single complex scalar. All covariant derivatives here and below are associated with the background magnetic field, and when necessary, it is also spatially covariantized as well.

To proceed, we act $D\star$ on the first equation of (2.22) and obtain

$$\begin{aligned} i\frac{n}{2} \text{vol}_{S^2} \wedge \delta A - \frac{n}{2} D\star \delta A &= -\frac{2\pi i}{k} (\bar{\phi}_0 D\star D\delta\phi^+ - d\star d\bar{\phi}_0 \delta\phi^+) \\ &= -\frac{2\pi i}{k} \bar{\phi}_0 \left(D\star D\delta\phi^+ + \frac{1}{4} \text{vol}_3 \delta\phi^+ \right) \end{aligned} \quad (2.24)$$

where we used $\star^2 = -1$, $d\star d\bar{\phi}_0 = -\frac{1}{4}\bar{\phi}_0\text{vol}_3$, and

$$(D^2\delta A)_{\mu\nu\rho} = 3!D_{[\mu}D_\nu\delta A_{\rho]} = \frac{3!}{2}[D_{[\mu}, D_{\nu]}\delta A_{\rho]} = -i\frac{3!}{2}F_{[\mu\nu}\delta A_{\rho]} = -i(F\wedge\delta A)_{\mu\nu\rho}. \quad (2.25)$$

Since $D\star D\delta\phi = -\text{vol}_3 D^\mu D_\mu\delta\phi$, $D\star\delta A = -\text{vol}_3 D^\mu\delta A_\mu$, one obtains

$$i\frac{n}{2}\delta A_0 + \frac{n}{2}D^\mu\delta A_\mu = \frac{2\pi i}{k}\bar{\phi}_0\left(D^\mu D_\mu\delta\phi^+ - \frac{1}{4}\delta\phi^+\right). \quad (2.26)$$

Multiplying ϕ_0 on both sides, one obtains

$$i\frac{n}{2}\phi_0\delta A_0 + \frac{n}{2}\phi_0 D^\mu\delta A_\mu = i\frac{n}{2}\left(D^\mu D_\mu\delta\phi^+ - \frac{1}{4}\delta\phi^+\right), \quad (2.27)$$

which is exactly the scalar equation of motion for $\delta\phi^+$ in (2.23). Similar manipulation with the second equation of (2.22) yields the $\delta\bar{\phi}^-$ equation in (2.23). From this finding, we conclude that it suffices for us to solve (2.22) only.

One can easily check the following gauge invariance of (2.22):

$$\begin{aligned} \delta A &\mapsto \delta A + D\epsilon, & \delta\phi^+ &\mapsto \delta\phi^+ - i\phi_0\epsilon \\ \delta\tilde{A} &\mapsto \delta\tilde{A} + D\tilde{\epsilon}, & \delta\bar{\phi}^- &\mapsto \delta\bar{\phi}^- - i\bar{\phi}_0\tilde{\epsilon}. \end{aligned} \quad (2.28)$$

This comes from the *linearized* off-diagonal part of the $U(N)\times U(N)$ gauge transformation, which at this order does not change the background magnetic field. A convenient gauge is $\delta\phi^+ = 0$ and $\delta\bar{\phi}^- = 0$, which resembles the ‘unitary gauge’ in spontaneously broken gauge theories with Higgs fields. The resulting equation is

$$\left(\star D - \frac{n}{2}\right)\delta A = 0, \quad \left(\star D + \frac{n}{2}\right)\delta\tilde{A} = 0 \quad (2.29)$$

where D in both equations is $D = d - iA^{(0)}$.³ These equations (with ordinary derivative replacing D) are known as odd dimensional self-duality equations [20], which find their natural appearances in gauged supergravity theories.

We solve these equations by expanding with monopole vector spherical harmonics. This is most easily done by reformulating the problem on \mathbb{R}^3 , after *formally* defining Euclidean-like variable $\tau \equiv it$. (τ is imaginary below.) One obtains

$$\star D\Psi = -i\frac{n}{2}\Psi, \quad \star D\tilde{\Psi} = i\frac{n}{2}\tilde{\Psi} \quad (2.30)$$

where $\Psi, \tilde{\Psi}$ are $\delta A, \delta\tilde{A}$ on Euclidean $S^2 \times \mathbb{R}$: $(\Psi)_\tau = -i(\delta A)_t$, $(\Psi)_{\theta,\phi} = (\delta A)_{\theta,\phi}$, etc. Defining $r = e^\tau$ and rescaling fields with $\frac{1}{r}$, namely $\Psi_r = \frac{1}{r}\Psi_\tau$ (while $\Psi_{\theta,\phi}$ remain the same), one obtains

$$\vec{\nabla} \times \vec{\Psi} = -i\frac{n}{2r}\vec{\Psi}, \quad \vec{\nabla} \times \vec{\tilde{\Psi}} = -i\frac{n}{2r}\vec{\tilde{\Psi}} \quad (2.31)$$

³If one keeps the parameter R with dimensionless coordinates τ, θ, ϕ and the metric $ds_{\mathbb{R}^3 \times S^2}^2$ in (2.7), the coefficients $\mp\frac{n}{2}$ are replaced by $\mp\frac{n}{2R}$, since \star on p-form involves $\sqrt{g}g^{\mu_1\nu_1}\dots g^{\mu_p\nu_p} \sim R^{3-2p}$. The last factor guarantees that \star^2 is dimensionless.

on \mathbb{R}^3 . Actually this is the same expression as that appearing in the computation of superconformal index in [9]. There the 1-loop determinant $\vec{D} \times \delta \vec{A} - i[\sigma, \delta \vec{A}] - \vec{D} \delta \sigma$ over the bosonic part of vector multiplet is computed. The gauge chosen in [9] was the Coulomb gauge, but the differential operator becomes the same if one chooses $\delta \sigma = 0$ gauge instead.

We consider the two equations together below, where the upper/lower sign denotes the case with $\Psi, \tilde{\Psi}$, respectively. We look for configurations on $S^2 \times \mathbb{R}$ with frequency ω and angular momentum j , given by

$$\vec{\Psi}, \vec{\tilde{\Psi}} = \frac{1}{r\omega} \left(a_+ \vec{C}_{jm}^+ + a_- \vec{C}_{jm}^- + a_0 \vec{C}_{jm}^0 \right). \quad (2.32)$$

\vec{C}_{jm}^λ are monopole vector spherical harmonics with n units of flux on S^2 . $\lambda = +1, 0, -1$ for $j \geq \frac{|n|}{2} + 1$, and $\lambda = +1, 0$ for $j = \frac{|n|}{2}$. Finally, $\lambda = +1$ for $j = \frac{|n|}{2} - 1$. See appendix B.2 of [9]. Note that $\frac{1}{r\omega} = e^{-\omega\tau} = e^{-i\omega t}$ is the energy factor. For $j \geq \frac{|n|}{2} + 1$, one finds

$$\begin{pmatrix} \omega \pm \frac{n}{2} & 0 & s_+ \\ 0 & \omega \mp \frac{n}{2} & s_- \\ s_+ & -s_- & \mp \frac{n}{2} \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \\ a_0 \end{pmatrix} = 0, \quad (2.33)$$

where $s_\pm = \sqrt{\frac{j(j+1)-q^2 \pm q}{2}}$ with $q = \frac{n}{2}$, and the upper and lower sign is for Ψ and $\tilde{\Psi}$, respectively. Nonzero solution exists when the determinant of 3×3 matrix is zero,

$$\frac{n}{2} \left[\left(\omega \pm \frac{1}{2} \right)^2 - \left(j + \frac{1}{2} \right)^2 \right] = 0. \quad (2.34)$$

There are two independent solutions for Ψ with $\omega = -(j+1)$ and j , and for $\tilde{\Psi}$ with $\omega = -j$ and $j+1$. For $j = \frac{n}{2}$, we lose the mode \vec{C}^- . Also in this case $s_- = 0$ and $s_+ = \sqrt{q}$. The equations for $\vec{\Psi}, \vec{\tilde{\Psi}}$ are

$$\begin{pmatrix} \omega \pm \frac{n}{2} & s_+ \\ s_+ & \mp \frac{n}{2} \end{pmatrix} \begin{pmatrix} a_+ \\ a_0 \end{pmatrix} = 0. \quad (2.35)$$

The solution exists when $\omega = \mp(\frac{n}{2} + 1) = \mp(j+1)$. Finally, for $j = \frac{n}{2} - 1$ (when $n \geq 2$), only the mode \vec{C}^+ remains. The equation reduces to

$$\left(\omega \pm \frac{n}{2} \right) a_+ = 0, \quad (2.36)$$

which has solution if $\omega = \mp \frac{n}{2} = \mp(j+1)$.

Note that, in our gauge which simplified the analysis, the parallel modes $\delta\phi$ and the orthogonal modes $\delta\varphi$ apparently look different. We would now like to rewrite the solutions such that the background-dependence of $\delta A_\mu, \delta \tilde{A}_\mu, \delta B_a$ fluctuations can be addressed in a simple manner. To this end, we try to gauge transform the modes δA_μ and $\delta \tilde{A}_\mu$ back to $\delta\phi$ using (2.28) and make the latter look similar to the $\delta\varphi$ modes. Certainly this is possible for all modes with $j \geq \frac{n}{2}$ since there are corresponding $\epsilon, \tilde{\epsilon}$ scalar modes which do this job.⁴ The modes from δA_μ with $j \geq \frac{n}{2} + 1$, having frequency $\omega = -(j+1), j$ go to $\delta\phi^+ = -i\phi_0\epsilon$

⁴As summarized in [9], action of D does not change the values of j, m or the frequency ω in this case.

with frequency $\omega = \mp (j + \frac{1}{2})$ due to the multiplication of ϕ_0 . Similarly, the modes from $\delta\tilde{A}_\mu$ go to $\delta\bar{\phi}^-$ with frequencies $\omega = \pm (j + \frac{1}{2})$. This is identical to the spectrum of $\delta\varphi$ with $j \geq \frac{n}{2} + 1$. For $j = \frac{n}{2}$, one needs modes with $\omega = \pm (\frac{n}{2} + 1)$ in $\delta\phi^\pm$ to match the spectrum of $\delta\varphi$, which would translate to the modes in δA_μ with frequency $\omega = \frac{n}{2}, -\frac{n}{2} - 1$ and $\delta\tilde{A}_\mu$ with frequency $\omega = -\frac{n}{2}, \frac{n}{2} + 1$. However, from our analysis in (2.35), only the latter frequencies for $\delta A_\mu, \delta\tilde{A}_\mu$ exist in the spectrum. We understand it as considering δB_a having universal spectrum (namely, $\omega = \pm (j + \frac{1}{2})$ for $j \geq \frac{n}{2}$) and regard the lacking modes in $\delta\phi$ as constraints

$$[\delta\phi]_{j=\frac{n}{2}, \omega=j+\frac{1}{2}} \sim \bar{b}^a [\delta B_a]_{j=\frac{n}{2}, \omega=j+\frac{1}{2}} = 0 \quad (2.37)$$

for all $1i$ and $i1$ components of δB_a . Since the background \bar{b}^a carries charges under $U(1) \times U(1) \subset U(N) \times U(N)$, one should understand this constraint either as the $1i$ component in the adjoint of first $U(N)$, or the $i1$ component in the adjoint of second $U(N)$. Since the frequency is positive, the constrained creation operator would be its conjugate after quantization. Finally, the modes with $j = \frac{n}{2} - 1$ in $\delta A_\mu, \delta\tilde{A}_\mu$ cannot be gauge-transformed to scalars, since they are genuinely vector-like modes. They remain as the modes in $1i$ components of each $U(N)$.

To finish the analysis of classical solutions, we consider the off-diagonal fermions. For $\psi_{a\alpha}$, in its $1i$ and $i1$ components denoted by ψ_a^+ and ψ_a^- , the equation of motion is given by

$$-i\gamma^\mu D_\mu \psi_a^\pm \pm i\frac{n}{2} \psi_a^\pm = 0. \quad (2.38)$$

The last term comes from the coupling to $\sigma, \tilde{\sigma}$ background like [9]. The spatial part of the differential operator in this equation is the same as that appearing in appendix B.1 of [9]. There are modes with frequencies $\omega = \pm (j + \frac{1}{2})$ for $j \geq \frac{n+1}{2}$, and furthermore modes with $\omega = j + \frac{1}{2}$ for $j = \frac{n-1}{2}$. The last modes provide creation operators for $\bar{\psi}^a$.

To obtain the χ_a equation, one should also study terms obtained by integrating out $\lambda_\alpha, \tilde{\lambda}_\alpha$:

$$-\frac{4\pi i}{k} \text{tr} \left[(\psi_a \bar{A}^a - \bar{B}^a \chi_a) (A_b \bar{\psi}^b - \bar{\chi}^b B_b) \right] + \frac{4\pi i}{k} \text{tr} \left[(\bar{A}^a \psi_a - \chi_a \bar{B}^a) (\bar{\psi}^b A_b - B_b \bar{\chi}^b) \right]. \quad (2.39)$$

Denoting by χ_a^\pm the fluctuations in $1i$ and $i1$ components of first and second $U(N)$ group, respectively, the equation of motion is given by

$$-i\gamma^\mu D_\mu \chi_a^+ - i\frac{n}{2} \chi_a^+ + \frac{4\pi i}{k} \chi_b^+ \bar{b}^b b_a = 0, \quad -i\gamma^\mu D_\mu \chi_a^- + i\frac{n}{2} \chi_a^- - \frac{4\pi i}{k} b_a \bar{b}^b \chi_b^- = 0. \quad (2.40)$$

We decompose the fermions as (recall $b_a = \sqrt{\frac{kn}{4\pi}} z_a$)

$$\chi_a^\pm = z_a \xi^\pm + \epsilon_{ab} \bar{z}^b \zeta^\pm. \quad (2.41)$$

The Dirac equations for ξ^\pm and ζ^\pm are

$$-i\gamma^\mu D_\mu \xi^\pm \pm i\frac{n}{2} \xi^\pm = 0, \quad -i\gamma^\mu D_\mu \zeta^\pm \mp i\frac{n}{2} \zeta^\pm = 0. \quad (2.42)$$

The spectrum of ζ^\pm is same as that of ψ_a above, since the differential operator in the second equation of (2.42) is the same as that for χ_a in [9]. However, the sign of the second

creation operator	nature	ϵ	h_3	$\epsilon - h_3 - j_3$	(indices) _{group}
$(\delta A_\mu)_{j=\frac{n-1}{2}}$ $(\delta \tilde{A}_\mu)_{j=\frac{n-1}{2}}^*$	bosonic states	$\frac{n}{2}$	0	$j - j_3 + 1$	$(1i)_1$ $(i1)_2$
$b_a(\delta \bar{B}^a)_{j=\frac{n}{2}}$	bosonic constraints	$\frac{n}{2}$	0	$j - j_3$	$(i1)_1$ $(1i)_2$
$\bar{b}^a(\chi_a)_{j=\frac{n-1}{2}}$	fermionic states	$\frac{n+1}{2}$	1	$j - j_3$	$(1i)_1$ $(i1)_2$
$b_a(\bar{\chi}^a)_{j=\frac{n-1}{2}}$	fermionic constraints	$\frac{n-1}{2}$	-1	$j - j_3 + 1$	$(i1)_1$ $(1i)_2$

Table 2. Adjoint states and constraints ($\omega < 0$, creation operators).

term in ξ^\pm equation is flipped from that in [9], due to the the last terms in (2.40). The modes with $\omega = \pm(j + \frac{1}{2})$ for $j \geq \frac{n+1}{2}$ remain the same, but the modes for $j = \frac{n-1}{2}$ come with negative frequency $\omega = -(j + \frac{1}{2}) = -\frac{n}{2}$ due to this sign change. The last modes create $\bar{\xi}^\pm$ instead of ξ^\pm .

We would again like to express the spectrum of ξ^\pm, ζ^\pm in a way such that background dependence is addressed simply. We add by hand modes in ξ^\pm with $j = \frac{n-1}{2}$ and frequency $\omega = j + \frac{1}{2}$, just like ζ^\pm . Then the whole spectrum of χ_a is the same as that obtained in [9]. The additional mode we inserted is eliminated by the following fermionic constraint

$$[\xi^\pm]_{j=\frac{n-1}{2}, \omega=j+\frac{1}{2}} \sim \bar{b}^a [\chi_a^\pm]_{j=\frac{n-1}{2}, \omega=j+\frac{1}{2}} = 0. \tag{2.43}$$

The constraint is either in the $1i$ or $i1$ component, which is regarded as adjoint components in the first and second $U(N)$ gauge group, respectively, due to the multiplication of \bar{b}^a . Finally, there are left-over modes $\sim \bar{b}^a \chi_a^\pm$ with $j = \frac{n-1}{2}$ and $\omega = -(j + \frac{1}{2}) = -\frac{n}{2}$ in the $1i$ of first $U(N)$ and $i1$ of second $U(N)$, respectively, from the analysis in the above parenthesis.

Table 2 summarizes adjoint modes and constraints. As for the indices, $(1i)_1$ denotes the $1i$ 'th component in the adjoint representation of the first gauge group (subscript), for instance.

2.2 Quantization and the superconformal index

Quantizing the modes obtained in the previous subsection at weak-coupling $\frac{N}{k} \rightarrow 0$, including the background modes b_a , one can identify the Fock space. From this one can compute the partition function. Although the last computation is straightforward, in this paper we only compute the superconformal index.

In the quantum theory (at weak coupling), one can compute the symplectic 2-form on the phase space to obtain normalized oscillators. We do not do this explicitly here, but simply regard the modes with positive/negative frequencies as annihilation/creation operators, respectively.⁵ The background variables b_a are regarded as annihilation operators.

⁵However, we should mention the modes from gauge fields in table 2, since it is not clear *a priori* if $\omega \geq 0$ corresponds to annihilation/creation operators for these modes. The symplectic form is proportional to

$$-\frac{k}{4\pi} \sqrt{g_{S^2}} \epsilon^{\mu\nu} (\delta A_\mu \wedge \delta A_\nu - \delta \tilde{A}_\mu \wedge \delta \tilde{A}_\nu),$$

with opposite signs for two gauge fields. From the definition of $\vec{C}_{jm}^{\pm 1} \sim (\vec{D} + i\hat{r} \times \vec{D})Y_{jm}$ given in [9], one can show that the modes from δA_μ with $\omega < 0$ are indeed creations, while that from $\delta \tilde{A}_\mu$ with $\omega > 0$ are annihilations.

The Gauss' law (2.14) on the background now constrains the summation of the occupation numbers by

$$kn = b_a^\dagger b_a \tag{2.44}$$

after correct normalization. However, all other small fluctuations now participate on the right hand side of this equation as well. The right hand side is the generator of the global part of the $U(1) \times U(1) \subset U(N) \times U(N)$ gauge transformation. Since only the difference of two $U(1)$ couples nontrivially to matters, it may be viewed either as the generator of first $U(1)$ or minus the generator of second $U(1)$. The presence of kn on the left hand side implies that the $U(1)$ gauge singlet condition should be imposed in the presence of the background $U(1)$ charge, which is $-kn$ or $+kn$ from the viewpoint of the first/second gauge group, respectively. The gauge invariance condition for $U(N-1) \times U(N-1)$ is applied in the standard way. To impose these conditions later, it is convenient to introduce $2N$ chemical potentials $\alpha_i, \tilde{\alpha}_i$ ($i = 1, 2, \dots, N$) for the $U(1)^N \times U(1)^N \subset U(N) \times U(N)$ color charges. Our convention is that states with positive charges are weighted by positive powers of $e^{-i\alpha_i}$.

The partition function of the free theory is easily calculated by first considering the single particle partition function, namely that over the modes. We define it by

$$z(x, x', y_1, y_2, \zeta, \alpha, \tilde{\alpha}) = \text{tr} \left[x^{\epsilon+j_3} (x')^{\epsilon-h_3-j_3} y_1^{h_1} y_2^{h_2} \zeta^{2j_3} e^{-i \sum_{i=1}^N (\alpha_i q_i + \tilde{\alpha}_i \tilde{q}_i)} \right] \tag{2.45}$$

where q_i, \tilde{q}_i are $U(1)^N \times U(1)^N$ charges. The one particle index is obtained by setting $\zeta = -1$:

$$f(x, y_1, y_2, \alpha, \tilde{\alpha}) = z(x, x', y_1, y_2, -1, \alpha, \tilde{\alpha}) = \text{tr} \left[(-1)^F x^{\epsilon+j_3} y_1^{h_1} y_2^{h_2} e^{-i \sum_{i=1}^N (\alpha_i q_i + \tilde{\alpha}_i \tilde{q}_i)} \right]. \tag{2.46}$$

The dependence on chemical potential x' disappears in this limit. These quantities are also called partition functions or index over 'letters.'

Let us compute the contribution to f from various modes. Firstly, the modes in 11 or ij (for $i, j \neq 1$) of $U(N) \times U(N)$ are basically the same as those of conventional free field theory, considered in [14]. The one particle index from bi-fundamental modes is given by

$$e^{-i(\alpha_1 - \tilde{\alpha}_1)} f^+(x, y_1, y_2) + \sum_{i,j=2}^N e^{-i(\alpha_i - \tilde{\alpha}_j)} f^+(x, y_1, y_2) \tag{2.47}$$

where

$$f^+ = \left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}} \right) \frac{x^{\frac{1}{2}}}{1-x^2} - \left(\sqrt{y_1 y_2} + \sqrt{\frac{1}{y_1 y_2}} \right) \frac{x^{\frac{3}{2}}}{1-x^2}, \tag{2.48}$$

and the anti bi-fundamental index is given by

$$e^{i(\alpha_1 - \tilde{\alpha}_1)} f^-(x, y_1, y_2) + \sum_{i,j=2}^N e^{i(\alpha_i - \tilde{\alpha}_j)} f^-(x, y_1, y_2) \tag{2.49}$$

where

$$f^- = \left(\sqrt{y_1 y_2} + \sqrt{\frac{1}{y_1 y_2}} \right) \frac{x^{\frac{1}{2}}}{1-x^2} - \left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}} \right) \frac{x^{\frac{3}{2}}}{1-x^2}, \tag{2.50}$$

as explained in [14].

To evaluate the contribution to the single particle index from $1i$, $i1$ of $U(N) \times U(N)$ (where $i \neq 1$), we follow the decomposition of charged modes in the previous subsection, into bi-fundamentals and adjoints. The bifundamental/anti-bifundamental part of the index is simply given by that of [9], since all the modes are same in two cases. This is given by

$$\sum_{i=2}^N \left[e^{-i(\alpha_1 - \tilde{\alpha}_i)} x^n f^+(x, y_1, y_2) + e^{i(\alpha_1 - \tilde{\alpha}_i)} x^n f^-(x, y_1, y_2) \right] \quad (2.51)$$

plus

$$\sum_{i=2}^N \left[e^{-i(\alpha_i - \tilde{\alpha}_1)} x^n f^+(x, y_1, y_2) + e^{i(\alpha_i - \tilde{\alpha}_1)} x^n f^-(x, y_1, y_2) \right]. \quad (2.52)$$

Finally, we sum over the finite number of adjoint modes which are either in $1i$ or $i1$ components of one of the two $U(N)$ gauge groups. The necessary information is summarized in table 2. One finds the following indices (bosonic/fermionic constraints appear with \mp signs, respectively)

$$\begin{aligned} \text{gauge fields: } & \sum_{i=2}^N x x' [(x')^{n-2} + (x')^{n-3} x + \dots + x^{n-2}] \left(e^{-i(\alpha_1 - \alpha_i)} + e^{-i(\tilde{\alpha}_i - \tilde{\alpha}_1)} \right) \\ \text{bosonic constraint: } & - \sum_{i=2}^N [(x')^n + (x')^{n-1} x + \dots + x^n] \left(e^{-i(\alpha_i - \alpha_1)} + e^{-i(\tilde{\alpha}_1 - \tilde{\alpha}_i)} \right) \\ \text{fermionic states: } & - \sum_{i=2}^N x [(x')^{n-1} + (x')^{n-2} x + \dots + x^{n-1}] \left(e^{-i(\alpha_1 - \alpha_i)} + e^{-i(\tilde{\alpha}_i - \tilde{\alpha}_1)} \right) \\ \text{fermionic constraint: } & \sum_{i=2}^N x' [(x')^{n-1} + (x')^{n-2} x + \dots + x^{n-1}] \left(e^{-i(\alpha_i - \alpha_1)} + e^{-i(\tilde{\alpha}_1 - \tilde{\alpha}_i)} \right). \end{aligned} \quad (2.53)$$

Adding the first/third lines, and also the second/fourth lines, one observes a vast cancellation. The final answer for the adjoint single particle index is

$$f^{\text{adj}}(x, \alpha, \tilde{\alpha}) = -x^n \sum_{i=2}^N \left[e^{-i(\alpha_1 - \alpha_i)} + e^{-i(\tilde{\alpha}_i - \tilde{\alpha}_1)} + e^{-i(\alpha_i - \alpha_1)} + e^{-i(\tilde{\alpha}_1 - \tilde{\alpha}_i)} \right], \quad (2.54)$$

which is in perfect agreement with the result of [9].

Let us define the sum of (2.47), (2.49), (2.51), (2.52) to be $f^{\text{matter}}(x, y_1, y_2, \alpha, \tilde{\alpha})$. The full index is obtained from one particle index by the multi-particle (or Plethystic) exponential of $f^{\text{matter}} + f^{\text{adj}}$. At this point we impose the $U(1)$ and $U(N-1) \times U(N-1)$ singlet conditions with background $U(1)$ charge. The final result is

$$\begin{aligned} I(x, y_1, y_2) = & \int \frac{d\alpha_1 d\tilde{\alpha}_1}{(2\pi)^2} e^{ikn(\alpha_1 - \tilde{\alpha}_1)} \frac{1}{[(N-1)!]^2} \prod_{i=2}^N \left[\frac{d\alpha_i d\tilde{\alpha}_i}{(2\pi)^2} \right] \prod_{i < j} \left[2 \sin \frac{\alpha_i - \alpha_j}{2} \right]^2 \left[2 \sin \frac{\tilde{\alpha}_i - \tilde{\alpha}_j}{2} \right]^2 \\ & \times \exp \left[\sum_{p=1}^{\infty} \frac{1}{p} \left(f^{\text{matter}}(x^p, y_1^p, y_2^p, p\alpha, p\tilde{\alpha}) + f^{\text{adj}}(x^p, p\alpha, p\tilde{\alpha}) \right) \right]. \end{aligned} \quad (2.55)$$

This was used in [9] to reproduce the large N supergravity index.

2.3 Generalizations and open problems

One can try to generalize the analysis in the previous two subsections. Firstly, one can take more magnetic charges to be nonzero $n_i, \tilde{n}_i \neq 0$ for some $i \geq 2$, while still satisfying $H = \tilde{H}$. We discuss some aspects of the semi-classical analysis and point out some puzzles, or subtleties. Secondly, one can consider the case $H \neq \tilde{H}$, which is claimed to be present during comparison with supergravity [9]. We have very little to say about this case here, apart from modest comments at the end of this subsection.

We first consider the case in which $n_i = \tilde{n}_i$ for $i = 1, 2, \dots, N$ where more than one pairs of fluxes are nonzero. For simplicity, we take all of them to be non-negative. Among $H = \{n_i\}$, some of them can be identical. If the fluxes are given by

$$\underbrace{p_1, p_1, \dots, p_1}_{N_1} > \underbrace{p_2, \dots, p_2}_{N_2} > \dots > \underbrace{p_f, \dots, p_f}_{N_f} \tag{2.56}$$

where $N_1 + N_2 + \dots + N_f = N$, the gauge symmetry $U(N) \times U(N)$ is broken to $\prod_{i=1}^f U(N_i) \times U(N_i)$. Again one can obtain classical solutions which will account for the lowest energy states. We take all $U(1)^N \times U(1)^N$ magnetic fluxes in (2.56) to be uniform on S^2 , set $A_t = 0$, and restrict the scalars $B_{\hat{a}}$ to be s-waves and also to be block-diagonal in $\prod_{i=1}^f U(N_i) \times U(N_i)$. We take

$$B_{\hat{a}} = b_{\hat{a}} e^{-it/2}, \tag{2.57}$$

where $b_{\hat{a}}$ are constant block-diagonal matrices. To solve the Gauss' law, they are subject to the following condition

$$\frac{kp_i}{4\pi} \mathbf{1}_{N_i} = \bar{b}^a b_a, \quad \frac{kp_i}{4\pi} \mathbf{1}_{N_i} = b_a \bar{b}^a \tag{2.58}$$

in each of the $U(N_i) \times U(N_i)$ block. Inserting the above relation to the scalar equation of motion (2.12), one finds that it is satisfied as well.

The constraint (2.58) from the Gauss' law may be solved as follows. In each block-diagonal sector, the matrices b_a can be diagonalized with unitary matrices U_a, V_a as

$$b_a = U_a^\dagger \mathcal{D}_a V_a. \tag{2.59}$$

The constraints can then be written in the i 'th block as

$$U_2^\dagger \mathcal{D}_2 \mathcal{D}_2^\dagger U_2 = U_1^\dagger \left(\frac{kp_i}{4\pi} - \mathcal{D}_1 \mathcal{D}_1^\dagger \right) U_1, \quad V_2^\dagger \mathcal{D}_2^\dagger \mathcal{D}_2 V_2 = V_1^\dagger \left(\frac{kp_i}{4\pi} - \mathcal{D}_1^\dagger \mathcal{D}_1 \right) V_1. \tag{2.60}$$

One finds that $U_2 = S_{N_i} U_1, V_2 = S_{N_i} V_1$, where S_{N_i} denotes an element of permutation of eigenvalues. Thus b_1 and b_2 can be simultaneously diagonalized by $U(N_i) \times U(N_i)$. Denoting the eigenvalues of \mathcal{D}_a by λ_{an} ($n = 1, 2, \dots, N_i$), one obtains

$$|\lambda_{1n}|^2 + |\lambda_{2n}|^2 = \frac{kp_i}{4\pi}. \tag{2.61}$$

There still remains S_{N_i} permutation symmetry acting on N_i doublets $(\lambda_{1n}, \lambda_{2n})$ of eigenvalues.

Before discussing the excitations from this solution, we first try to quantize the lowest energy states. One can do this in two approaches. One can either start from the diagonalized variables (2.61) with permutation symmetry, or work directly with block-diagonal variables satisfying (2.14). The first way is essentially the quantization of moduli space discussed in, say, [4].

We start by considering the first method. Each pair of eigenvalues (λ_1, λ_2) , correctly normalized as creation/annihilation operators after quantization, are constrained to satisfy $\lambda_1^\dagger \lambda_1 + \lambda_2^\dagger \lambda_2 = p_i k$, constraining the sum of occupation numbers to be $p_i k$. The single particle partition function is given by (y is conjugate to SU(2) Cartan)

$$z(x, y) = x^{\frac{p_i k}{2}} \left(y^{p_i k} + y^{p_i k - 2} + \dots + y^{-p_i k} \right). \quad (2.62)$$

The full N_i -particle partition function $Z_{N_i}(x, y)$ in i 'th block is given by

$$\begin{aligned} Z(x, y, \nu) &\equiv \sum_{N=0}^{\infty} \nu^N Z_N(x, y) = \exp \left(\sum_{n=1}^{\infty} \frac{\nu^n}{n} \zeta(x^n, y^n) \right) \\ &= \frac{1}{\left(1 - \nu x^{\frac{p_i k}{2}} y^{p_i k} \right) \left(1 - \nu x^{\frac{p_i k}{2}} y^{p_i k - 2} \right) \dots \left(1 - \nu x^{\frac{p_i k}{2}} y^{-p_i k} \right)}. \end{aligned} \quad (2.63)$$

Actually this is the dual graviton partition function at lowest energy. The full partition function is given by the product of partition functions for all blocks.

We can also quantize this sector by working with $2N^2$ matrix elements of oscillators $(b_a)^\dagger$. For the mn 'th matrix element of $(b_a)^\dagger$, the partition function for the single oscillator is

$$\frac{1}{1 - x^{\frac{1}{2}} y^{\pm 1} e^{-i(\alpha_m - \tilde{\alpha}_n)}} \quad (2.64)$$

where ± 1 is for $a = 1, 2$, respectively. $\alpha_n, \tilde{\alpha}_n$ are $2N$ chemical potentials conjugate to the $U(N_i) \times U(N_i)$ color charges. The quantum version of the constraints (2.58) are given by

$$p_i k \mathbf{1}_{N_i} =: b_a b_a^\dagger :, \quad p_i k \mathbf{1}_{N_i} =: b_a^\dagger b_a : \quad (2.65)$$

where $::$ denotes normal ordering. The right hand sides of the two equations are minus the generator of the second $U(N)$ gauge transformation, and the generator of the first $U(N)$ gauge transformation, respectively. The presence of left hand sides imply that gauge invariance has to be imposed with the background electric charges $(-p_i k \mathbf{1}_{N_i}, p_i k \mathbf{1}_{N_i})$ for $U(N_i) \times U(N_i)$. Collecting all, the partition function counting gauge invariant states is given by

$$Z_{N_i}(x, y) = \int [dU][d\tilde{U}] e^{i k p_i \sum_{n=1}^{N_i} (\alpha_n - \tilde{\alpha}_n)} \prod_{m,n=1}^{N_i} \frac{1}{\left(1 - x^{\frac{1}{2}} y e^{-i(\alpha_m - \tilde{\alpha}_n)} \right) \left(1 - x^{\frac{1}{2}} y^{-1} e^{-i(\alpha_m - \tilde{\alpha}_n)} \right)} \quad (2.66)$$

where the phase factor is due to the background charge, and the unitary matrices U, \tilde{U} can be diagonalized with eigenvalues $\{e^{i\alpha_n}\}, \{e^{i\tilde{\alpha}_n}\}$. The Haar measure appearing in the integral is

$$[dU][d\tilde{U}] = \frac{1}{(N_i!)^2} \left[\frac{d\alpha_n d\tilde{\alpha}_n}{(2\pi^2)} \right] \prod_{m < n} \left(2 \sin \frac{\alpha_m - \alpha_n}{2} \right)^2 \left(2 \sin \frac{\tilde{\alpha}_m - \tilde{\alpha}_n}{2} \right)^2. \quad (2.67)$$

This is the same as the field theory index in [9], which is actually the partition function since there are no fermions. Again, the full partition function is given by the product for all blocks.

We turn to discuss the fluctuations of modes around this background in the leading order in $\frac{1}{k}$. Contrary to the analysis in previous sections, we encounter a subtlety starting from the simplest fluctuations δA_a . We will simply point this out in this case, leaving a detailed study for the future.

We take the diagonalized background scalars satisfying (2.61). Let us consider the fluctuations δA_a in bifundamental of $U(N_1) \times U(N_2)$.⁶ After an analysis similar to section 2.1, the kinetic and mass terms become

$$\frac{d\delta A_a}{dt} \frac{d\delta \bar{A}^a}{dt} - \left[\left(j + \frac{1}{2} \right)^2 - \frac{(p_1 - p_2)^2}{4} \right] \delta A_a \delta \bar{A}^a \quad (2.68)$$

where $j \geq \frac{|p_1 - p_2|}{2}$. Its appearance in the potential can also be analyzed similarly. From the coupling of A_a to $\sigma, \tilde{\sigma}$ in (2.3), one obtains

$$- \frac{(p_1 - p_2)^2}{4} \delta A_a \delta \bar{A}^a \quad (2.69)$$

in the leading order. One should also consider the potential coming from the superpotential in this case: from (2.17), one obtains (superscripts in $\lambda_{am}^1, \lambda_{bn}^2$ refer to first/second blocks)

$$- \left| \epsilon^{ab} \lambda_{am}^1 \lambda_{bn}^2 \right|^2 \delta A_{amn} (\delta A_{amn})^* \quad (2.70)$$

where $m = 1, 2, \dots, N_1$ and $n = 1, 2, \dots, N_2$. This is nonzero unless λ_{am}^1 and λ_{an}^2 are proportional as $SU(2)$ doublets. Collecting all, one obtains the classical modes of $(\delta A_a)_{mn}$ with the following frequency:

$$\omega^2 = \left(j + \frac{1}{2} \right)^2 + |\epsilon^{ab} \lambda_{am}^1 \lambda_{bn}^2|^2. \quad (2.71)$$

Apparently, the supersymmetric modes, which should satisfy $\omega \pm (j + \frac{1}{2})$, seem to be allowed only for specific backgrounds. It is not yet clear to us how to deal with these modes and, in particular, address the results in [9] from our approach.

The analysis in [9] also demands the existence of monopole operators with $H \neq \tilde{H}$, for the index to agree with the index from supergravity. This case is much harder to study in our approach than examples above, since we even do not know an exact classical solution in this background. Note that our semi-classical consideration of the exact backgrounds in the previous subsections resembles the quantization of moduli space, which is generally useful in dealing with the modes in s-waves. From the analysis in [9], states including monopole operators with $H \neq \tilde{H}$ carry nonzero spatial angular momenta, $j_3 \neq 0$. Perhaps it may not be effective to consider this case with our approach.

⁶The case with δA_a within a single block can also be obtained if one replaces p_2, N_2 by p_1, N_1 below.

3 Concluding remarks

In this paper we studied the spectrum of local operators which involve magnetic monopole operators. We considered the semi-classical quantization of all excitations around the exact classical solution, where the latter accounts for protected chiral operators with lowest energy in the monopole background. We used our result to reproduce the superconformal index of [9] in the simplest monopole background, namely $H = \tilde{H} = (n, 0, 0, \dots, 0)$.

A motivation of this study was to demystify some results obtained in [9]. Since the calculation there involved deforming the theory (in a way that the index does not change), it was hard to see what is actually going on physically, despite all the quantitative agreement with supergravity reported there. A novel feature was the appearance of degrees of freedom in the adjoint representation of $U(N) \times U(N)$ in the presence of nonzero flux. The analysis of this paper shows that this has to do with the interaction between some matters and gauge fields. It technically comes from ‘exceptional’ low-lying spherical harmonics with nonzero monopoles, with total angular momentum $j = \frac{|n|-1}{2}$ for spinors and $j = \frac{|n|}{2}, \frac{|n|}{2} - 1$ for vectors.

Another finding is that the actual spectrum is subtler than that in the ‘deformed’ theory, although the difference is guaranteed not to affect the index. Knowing the actual Hilbert space in the weakly interacting regime, one can try to systematically develop the relevant perturbation theory to higher orders in $\frac{1}{k}$. This will in principle enable us to study the open strings connecting heavy $D0$ branes (with mass $\sim k$) from the gauge theory. For instance, it may be interesting to see if one can obtain a useful open spin chain description for macroscopic open strings ending on $D0$ branes. See also [16] for a study of open strings ending on membrane giant gravitons.

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References

- [1] J.H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078 [[hep-th/0411077](#)] [[SPIRES](#)].
- [2] J. Bagger and N. Lambert, *Modeling multiple $M2$'s*, *Phys. Rev. D* **75** (2007) 045020 [[hep-th/0611108](#)] [[SPIRES](#)]; *Gauge Symmetry and Supersymmetry of Multiple $M2$ -Branes*, *Phys. Rev. D* **77** (2008) 065008 [[arXiv:0711.0955](#)] [[SPIRES](#)]; *Comments On Multiple $M2$ -branes*, *JHEP* **02** (2008) 105 [[arXiv:0712.3738](#)] [[SPIRES](#)].
- [3] A. Gustavsson, *Algebraic structures on parallel $M2$ -branes*, *Nucl. Phys. B* **811** (2009) 66 [[arXiv:0709.1260](#)] [[SPIRES](#)].

- [4] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[SPIRES](#)].
- [5] V. Borokhov, A. Kapustin and X.-k. Wu, *Topological disorder operators in three-dimensional conformal field theory*, *JHEP* **11** (2002) 049 [[hep-th/0206054](#)] [[SPIRES](#)]; *Monopole operators and mirror symmetry in three dimensions*, *JHEP* **12** (2002) 044 [[hep-th/0207074](#)] [[SPIRES](#)];
 G. 't Hooft, *On the Phase Transition Towards Permanent Quark Confinement*, *Nucl. Phys.* **B 138** (1978) 1 [[SPIRES](#)];
 P. Goddard, J. Nuyts and D.I. Olive, *Gauge Theories and Magnetic Charge*, *Nucl. Phys.* **B 125** (1977) 1 [[SPIRES](#)];
 V. Borokhov, *Monopole operators in three-dimensional $N = 4$ SYM and mirror symmetry*, *JHEP* **03** (2004) 008 [[hep-th/0310254](#)] [[SPIRES](#)];
 A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev.* **D 74** (2006) 025005 [[hep-th/0501015](#)] [[SPIRES](#)];
 A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, [hep-th/0604151](#) [[SPIRES](#)].
- [6] D. Berenstein and D. Trancanelli, *Three-dimensional $N = 6$ SCFT's and their membrane dynamics*, *Phys. Rev.* **D 78** (2008) 106009 [[arXiv:0808.2503](#)] [[SPIRES](#)].
- [7] I. Klebanov, T. Klose and A. Murugan, *AdS_4/CFT_3 — Squashed, Stretched and Warped*, *JHEP* **03** (2009) 140 [[arXiv:0809.3773](#)] [[SPIRES](#)].
- [8] Y. Imamura, *Monopole operators in $N = 4$ Chern-Simons theories and wrapped M2-branes*, *Prog. Theor. Phys.* **121** (2009) 1173 [[arXiv:0902.4173](#)] [[SPIRES](#)].
- [9] S. Kim, *The complete superconformal index for $N = 6$ Chern-Simons theory*, *Nucl. Phys.* **B 821** (2009) 241 [[arXiv:0903.4172](#)] [[SPIRES](#)].
- [10] M.M. Sheikh-Jabbari and J. Simon, *On Half-BPS States of the ABJM Theory*, *JHEP* **08** (2009) 073 [[arXiv:0904.4605](#)] [[SPIRES](#)].
- [11] M.K. Benna, I.R. Klebanov and T. Klose, *Charges of Monopole Operators in Chern-Simons Yang-Mills Theory*, [arXiv:0906.3008](#) [[SPIRES](#)].
- [12] A. Gustavsson and S.-J. Rey, *Enhanced $N = 8$ Supersymmetry of ABJM Theory on $R(8)$ and $R(8)/Z(2)$* , [arXiv:0906.3568](#) [[SPIRES](#)].
- [13] O.-K. Kwon, P. Oh and J. Sohn, *Notes on Supersymmetry Enhancement of ABJM Theory*, *JHEP* **08** (2009) 093 [[arXiv:0906.4333](#)] [[SPIRES](#)].
- [14] J. Bhattacharya and S. Minwalla, *Superconformal Indices for $\mathcal{N} = 6$ Chern Simons Theories*, *JHEP* **01** (2009) 014 [[arXiv:0806.3251](#)] [[SPIRES](#)].
- [15] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[SPIRES](#)];
 J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju, *Indices for Superconformal Field Theories in 3, 5 and 6 Dimensions*, *JHEP* **02** (2008) 064 [[arXiv:0801.1435](#)] [[SPIRES](#)];
 C. Romelsberger, *Counting chiral primaries in $N = 1$, $D = 4$ superconformal field theories*, *Nucl. Phys.* **B 747** (2006) 329 [[hep-th/0510060](#)] [[SPIRES](#)].
- [16] D. Berenstein and J. Park, *The BPS spectrum of monopole operators in ABJM: towards a field theory description of the giant torus*, [arXiv:0906.3817](#) [[SPIRES](#)].

- [17] M. Benna, I. Klebanov, T. Klose and M. Smedback, *Superconformal Chern-Simons Theories and AdS₄/CFT₃ Correspondence*, *JHEP* **09** (2008) 072 [[arXiv:0806.1519](#)] [[SPIRES](#)].
- [18] S. Bhattacharyya and S. Minwalla, *Supersymmetric states in M5/M2 CFTs*, *JHEP* **12** (2007) 004 [[hep-th/0702069](#)] [[SPIRES](#)];
L. Grant, P.A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 BPS Quantum States and Classical Configurations*, *JHEP* **05** (2008) 049 [[arXiv:0803.4183](#)] [[SPIRES](#)].
- [19] F.A. Dolan, *On Superconformal Characters and Partition Functions in Three Dimensions*, [arXiv:0811.2740](#) [[SPIRES](#)].
- [20] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Selfduality in Odd Dimensions*, *Phys. Lett.* **B 136** (1984) 38 [*Addendum ibid.* **B 137** (1984) 443] [[SPIRES](#)].